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ORIGINAL PAPER

Traveling pulses and its wave solution scheme in a diffusively coupled 2D Hindmarsh-Rose excitable systems

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Abstract In this article, an analytical approach is demonstrated to show the emerging traveling pulses for the local evolution of a set of diffusively coupled dynamical equations representing neuronal impulses. The derived dynamics governing the traveling pulses solution is described in a space-time reference frame with a two-dimensional excitable Hindmarsh-Rose (H-R) type oscillator. We deduce the conditions that allow us to describe explicitly the nature of propagating traveling pulses. We have constructed the detailed analytical results using semi-discrete approximation method with numerical simulations illuminating possible traveling pulses that include dispersion relations and group velocity equations. We show that the diffusive network can be expressed by the complex Ginzburg-Landau equation. The extended excitable medium with a homogeneous diffusive connection exhibits envelope solitons and multipulses. We observe how the series expansion parameter and coupling play key roles for the appearance of different traveling pulses. The transition phases and amplitude modulations are reported. The obtained results in the form of single solitary pulses and multipulses, reveal the possibility of collective behavior for information processing in excitable system.

Keywords H-R system · Diffusive network · Semi-discrete approximations · Traveling pulses

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1 Introduction

Diverse biophysically motivated excitable models have been studied to examine neuro-computational characteristics that produce various firing modes [8,16,18,23]. Single neuron models have the advantage to reproduce diverse firing properties [16,17]. These models are often used to examine the biophysical features of the nervous system using various network topologies in cortical regions [3,4,6,17,18] . Hindmarsh and Rose [14,15] introduced a simplified reduced biophysical model originated from the Hodgkin-Huxley type model and it exhibits various neuronal firing properties. Recently, the intrinsic dynamics of the excitable models have been studied, especially on H-R dynamics. The model is the extended modification of the FitzHugh-Nagumo (FHN) type model [9]. Realistic electrical activities of neurons in the H-R model can be reproduced by introducing a nonlinear polynomial function in the FHN dynamical model [22]. Often it is used to examine the neuro-computational characteristics using various network formations in cortical networks [16,17]. Tsuji et al. [40] described a 2D H-R type model exhibiting the properties of both Class I and Class II neurons with detailed bifurcation analysis and studied various firing features. Although the model contains only four parameters, the obtained bifurcation structure satisfies conditions for emergence of both excitability features with constant stimuli. Later Chen et al. [5] worked on the H-R dynamics using the spike-and-rest properties and established the existence of diverse Hopf bifurcation points.

In excitable spatial systems of coupled neuronal networks, a transmembrane voltage difference can travel across the nerve cells as propagating waves [21,26,35,37]. Hence, it may effect appearance and disappearance of emerging traveling waves as neuronal responses. Such types of behavior are responsible, that can play a major role in signal processing [1,3,7,11,13,20,37]. Traveling pulses show a characteristic feature that can be explored with the underlying mechanisms of wave propagation in neuronal tissue. It is a relevant study in understanding both normal and pathological conditions in neural dynamics [7,11,12,20,27,31,41,43]. The propagation of the synchronous activity and the effect of the network parameters including plasticity, network architecture, synaptic constants and inhomogeneity have been widely studied in the recent decades [2,6,32,36]. The propagation of wavy profiles in the diffusion induced networks can appear due to the effects of different electrical activities at the single cell level. The appearance of various traveling waves is often occurred due to self-organization and interactions in the coupled network. It might review the regular-irregular dynamics in signal processing, and functional mechanisms in neural system.

In this study, our approach deals with the appearance and disappearance of emerging diverse traveling waves in a spatially excitable biophysical media. The work is motivated by theoretical and experimental studies that report the presence of localized excitations in certain neuronal networks [7,19,29,39,41]. In our previous article [29], the existence of various single pulses in the H-R spiking-bursting model has been explored using the tanh method [25,42]. Further, a particular type of traveling waves known as the envelope of a multibump solution has been established in [20]. Recently, we have discussed a different analytical approach (semi-discrete approximation) to obtain single as well as multipulses for a slow-fast FitzHugh-Rinzel (FHR) model [28]. Particularly, here we investigate the local nonlinear excitations in the diffusively induced system that forms a coupled systems of partial differential equations (PDEs). The structural transitions of different wave profiles [4,7,11,20,29,33,43] of various wave forms can be illustrated using our methodology using a weakly nonlinear theory. We observe envelope solitons, multibump solutions and show how the predominant parameters with coupling strength play a major role in the formation of traveling wave profiles.

The underlying mechanisms for the emergence of traveling fronts and pulses have been investigated mathematically [19,28]. Biophysical models can generate two types of traveling pulses: single traveling pulses and multipulses. The appearance and disappearance of bumps in multibump solutions change the feature of wave propagation. However, it is interesting to test mathematically the existence of exact or approximate solutions to the spatial excitable systems for the emergence of traveling waves to establish the numerical results. Here, we describe the emergence of wave propagation, the stability of the solution and speed equations. Multiple equilibrium solutions can lead to two traveling pulses known as traveling fronts and pulses. A traveling front connects different steady states at both ends, whereas, a traveling pulse connects the same steady state [20,33].

The essential mechanisms are constructed using a diffusively coupled 2D H-R network with homogeneous connection, which exhibits spiking activity in the single cell dynamics. We demonstrate the multiple-scale expansion concept in semi-discrete approximation theory, then we obtain the modified complex Ginzburg-Landau equation (CGLE) using perturbation theory. We derive the phase and amplitude of the traveling waves. We show the effects of coupling strength on the formation of wave propagation as well as their characteristics. Finally, we observe the multipulses and the transition phases to solitary pulse for small perturbations on the solution of the wave profiles. We derive the expressions of the wave profiles using the voltage variable. In particular, we have constructed detailed analytical results with numerical simulations illuminating possible traveling pulses with dispersion relations and group velocity equations. The study of neuronal impulses that are propagated in the form of traveling pulses in such networks may be relevant in brain pathological dynamics for certain functional mechanisms in neural system [10,24,34,38].

2 The solution scheme

2.1 Description of the coupled network

We describe the analysis by considering a biophysically motivated 2D H-R type spiking model [5,22,30,40]. It can produce various neuro-computational features for both Class I & Class II excitabilities. The two system variables u and v in Eq. (1) represent the membrane voltage dynamics and a recovery variable, respectively. The positive parameters a, b, c and d have constant values. The parameter c denotes the time scale. The parameter I measures applied stimulus current. The goal is to describe the spatially extended system using the single deterministic model exhibiting oscillatory responses. The dynamical equations of the excitable network of $N(k = 1, 2, ..., N)$ oscillators with bidirectional electrical coupling are described as follows

$$\begin{aligned}\frac{du_k}{dt} &= c(u_k - \frac{u_k^3}{3} - v_k + I)\\ &\quad +D(u_{k+1} - 2u_k + u_{k-1}),\\ \frac{dv_k}{dt} &= (u_k^2 + du_k - bv_k + a)/c,\end{aligned} \tag{1}$$

where D indicates the coupling. We consider two nearest neighbor coupling configuration. The nodes in the network are connected to their nearest neighboring nodes by electrical coupling. The variability may consider the characteristics at various levels of specific neuronal receptors or differences in the regulatory effects which can be influenced by applied or internal neuromodulatory process [7,20,33]. In order to perform the analysis and study the interactions in the extended excitable network, the extended system is considered such that each oscillator in the network is represented in the domain of equispaced mesh with the same coupling topology in the homogeneous system. The dynamics of wave generation and propagation can be described by the dynamical H-R model. The large scale coupled excitable network can produce diverse spatial dynamics with different structures. One of these types of dynamics induce traveling pulses. First, we transform the spatial system by reducing two equations of model into one second-order differential equation. The transformation approach does not affect the behavior of the dynamical system, then the governing equation becomes

$$\begin{aligned}&\ddot{u}_k + [b/c - c + cu_k^2]\dot{u}_k + (d-b)u_k + u_k^2\\ &\quad + \frac{bu_k^3}{3} + a - Ib = D(\dot{u}_{k+1} - 2\dot{u}_k + \dot{u}_{k-1})\\ &\quad + bD(u_{k+1} - 2u_k + u_{k-1})/c.\end{aligned} \tag{2}$$

Mathematically, it is complicated to derive the explicit analytical solution of the extended nonlinear system, we can deduce the approximate solution using the semi discrete approximation method. By introducing a new variable $u_k = \epsilon m_k$, where $0 < \epsilon < 1$, then Eq. 2 becomes

$$\begin{aligned}&\ddot{m}_k + [(b/c - c) + c\epsilon^2 m_k^2]\dot{m}_k + (d-b)m_k + \epsilon m_k^2\\ &\quad + \epsilon^2 b\frac{m_k^3}{3} = D(\dot{m}_{k+1} - 2\dot{m}_k + \dot{m}_{k-1})\\ &\quad + bD(m_{k+1} - 2m_k + m_{k-1})/c.\end{aligned} \tag{3}$$

This method is also used to study the plane wave modulation that is caused by the nonlinear terms in the dynamics of the system [19,39]. To obtain the solution $u(x,t)$ in spatial coordinates, we introduce new space and time variables as follows: $X_k = \epsilon^k x$ and $T_k = \epsilon^k t$, it presents a perturbation series of functions. Now, we consider

$$u(x,t) = \textstyle\sum_{k=1}^{\infty} \epsilon^k m_k(X_0, X_1, X_2, ...T_0, T_1, T_2, ...). \tag{4}$$

The partial derivatives are described as follows

$$\begin{aligned}\tfrac{\partial}{\partial t} &= \tfrac{\partial}{\partial T_0} + \epsilon\tfrac{\partial}{\partial T_1} + \epsilon^2\tfrac{\partial}{\partial T_2} + ...\\ \tfrac{\partial}{\partial x} &= \tfrac{\partial}{\partial X_0} + \epsilon\tfrac{\partial}{\partial X_1} + \epsilon^2\tfrac{\partial}{\partial X_2} + ...\end{aligned} \tag{5}$$

An interesting property of this multiple scale expansion method is that, we can find the solution of the original problem if new multi-dimensional space comes from the physical line. To derive the expressions of the solution, we consider the membrane voltage variable in the space and time coordinates using the series of derivatives and compare different terms in diverse orders of ϵ.

3 Equations of motion of the network

We derive the solution both analytically and numerically for the diffusive induced network using multiple scale expansion approach using this semi discrete approximation theory. First, we transformed Eq. 1 in a second-order ODE (Eq. 3) using Eq. (2). Next, we perturb G_0 and D_0 upto order ϵ^2, then the system becomes

$$\ddot{m}_k + \epsilon^2[G_0 + cm_k^2]\dot{m}_k + (d-b)m_k + \epsilon m_k^2 + \epsilon^2 b\frac{m_k^3}{3} = \epsilon^2 D_0(\dot{m}_{k+1} - 2\dot{m}_k + \dot{m}_{k-1}) + bD(m_{k+1} - 2m_k + m_{k-1})/c, \quad (6)$$

where, $G_0 = b/c - c$ and $D_0 = D$. For simplicity, we have mentioned D_0 as D from here on. Now, we deal with the weakly coupled spatial network with nonlinear excitations. In order to obtain the modified CGLE equation, we apply the multiple scale expansion procedure. We assume the solution of Eq. 6 as follows

$$m_k = P_k e^{i\Omega_k} + P_k^* e^{-i\Omega_k} + \epsilon(Q_k + R_k e^{2i\Omega_k} + R_k^* e^{-2i\Omega_k}) + O(\epsilon^2) \quad (7)$$

where $\Omega_k = qk - \omega t$. q and ω denote normal mode wave vector and angular velocity [28]. The continuum limit approximation is described on the amplitudes $P_k(t)$, $Q_k(t)$ and $R_k(t)$ as they change slowly with respect to the space and time coordinates. Then, we use the continuum limit approximation theory on the amplitudes $P_k(t)$, $Q_k(t)$ and $R_k(t)$, it becomes $P(X_1, X_2, T_1, T_2)$, $Q(X_1, X_2, T_1, T_2)$ and $R(X_1, X_2, T_1, T_2)$. Using Taylor series expansion, $P_{k\pm1}$ is given by

$$P_{k\pm1} = P \pm \epsilon\frac{\partial P}{\partial X_1} \pm \epsilon^2\frac{\partial P}{\partial X_2} + \frac{\epsilon^2}{2}\frac{\partial^2 P}{\partial X_1^2} + O(\epsilon^3).$$

First- and second-order temporal derivatives are described as

$$\frac{\partial P_k}{\partial t} = \epsilon\frac{\partial P}{\partial T_1} + \epsilon^2\frac{\partial P}{\partial T_2} + o(\epsilon^3),$$

and

$$\frac{\partial^2 P_k}{\partial t^2} = \epsilon^2\frac{\partial^2 P}{\partial T_1^2} + o(\epsilon^3),$$

respectively. Similarly, we can find for the amplitudes P, Q and R. First- and second-order derivatives of m_k with new space and time variables and amplitudes P, Q, R are obtained by

$$\begin{aligned}\dot{m}_k = {} & (\epsilon\frac{\partial P}{\partial T_1} + \epsilon^2\frac{\partial P}{\partial T_2} - i\omega P)e^{i\Omega_k} \\ & + (\epsilon\frac{\partial P^*}{\partial T_1} + \epsilon^2\frac{\partial P^*}{\partial T_2} + i\omega P^*)e^{-i\Omega_k} \\ & + \epsilon^2\frac{\partial Q}{\partial T_1} + (\epsilon^2\frac{\partial R}{\partial T_1} - \epsilon 2i\omega R)e^{2i\Omega_k} \\ & + (\epsilon^2\frac{\partial R^*}{\partial T_1} + \epsilon 2i\omega R^*)e^{-2i\Omega_k} + o(\epsilon^3),\end{aligned}$$

$$\begin{aligned}\ddot{m}_k = {} & (\epsilon^2\frac{\partial^2 P}{\partial T_1^2} - \epsilon 2i\omega\frac{\partial P}{\partial T_1} - \epsilon^2 2i\omega\frac{\partial P}{\partial T_2} - \omega^2 P)e^{i\Omega_k} \\ & + (\epsilon^2\frac{\partial^2 P^*}{\partial T_1^2} + \epsilon 2i\omega\frac{\partial P^*}{\partial T_1} + \epsilon^2 2i\omega\frac{\partial P^*}{\partial T_2} \\ & - \omega^2 P^*)e^{-i\Omega_k} + (-\epsilon^2 4i\omega\frac{\partial R}{\partial T_1} - \epsilon 4\omega^2 R)e^{2i\Omega_k} \\ & + (\epsilon^2 4i\omega\frac{\partial R^*}{\partial T_1} - \epsilon 4\omega^2 R^*)e^{-2i\Omega_k} + o(\epsilon^3).\end{aligned}$$

Now, using aforementioned equations in Eq.6 and equating the coefficients of different orders of ϵ, we can find the following relations.

Note1: We get the dispersion relation of pulses for the considered diffusive network by comparing the coefficient of $e^{\pm i\Omega_k}$ in Eq.6 (see Appendix). The dispersion relation is given by

$$\omega^2 = (d-b) + (4bD\sin^2(q/2))/c. \quad (8)$$

We have discussed the dispersion relation in Fig. 1 (a) for various values of couplings, D (green, red and blue lines indicate for $D = 0.01, 1, 10$). In the dispersion relation of the pulses in the network, we observe that it depends on the parameters of the diffusively coupled system and with the increase of D, the angular velocity, ω changes.

Note 2: Similarly, we compare the coefficient of $\epsilon e^{i\Omega_k}$ in Eq. 6 (see Appendix), we get

$$\frac{\partial P}{\partial T_1} + v_g\frac{\partial P}{\partial X_1} = 0. \quad (9)$$

v_g indicates as group velocity, the relationship is given by

$$v_g = \frac{bD\sin q}{c\omega}. \quad (10)$$

Diffusive coupling, D plays a key role in controlling the velocity of the traveling pulses. We have discussed the effects of diffusion coefficient, D on the velocity of the traveling pulses (Fig. 1 b). At lower coupling, velocity v_g changes very rapidly, however at higher coupling, the rate of change of velocity is lower than previous one. The velocity of propagating waves depends on the diffusive nature of the membrane voltage dynamics, then the propagating waves move faster across the cell membrane with the ionic movements.

Now, we equate the coefficients involving no exponential term and $\epsilon e^{2i\Omega_k}$ in Eq. 6 (see Appendix), we find

$$Q = -\frac{2}{d-b}PP^* \quad (11)$$

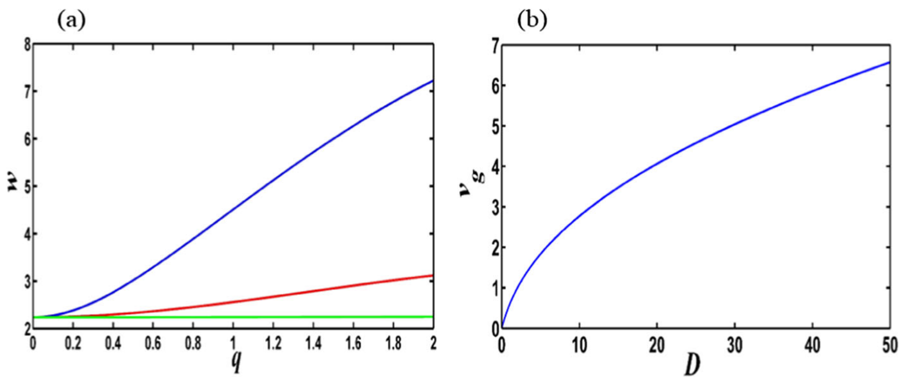

Fig. 1 The effects of normal mode wave vector, q and diffusive coupling, D on ω and the group velocity, v_g. Dispersion relation for the set of parameters [5,40] $c = 3$, $b = 5$, $d = 10$ and (a) for $D = 0.01, 1, 10$ indicated by green, red and blue lines respectively. (b) The effects of D on the velocity of the traveling pulses for $c = 3$, $b = 5$, $d = 10$ and $q = 1.5$, the variations of v_g show that it changes very rapidly, however at higher diffusive coupling, the rate of change is very low

and

$$R = \frac{P^2}{4\,\omega^2 - (d-b) - 4bD\left(\sin^2 q\right)/c} \tag{12}$$

respectively.

Result 1: The terms depending on $\epsilon^2 e^{i\Omega_k}$ describe the following relation for Eq. 6 (see Appendix)

$$\begin{aligned}\tfrac{\partial^2 P}{\partial T_1^2} - 2i\omega\tfrac{\partial P}{\partial T_2} &= i\omega(b/c - c)P + (i\omega c - b)\mid P|^2 P \\ &\quad -2(PQ + P^* R) + 4i\omega P D \sin^2 \tfrac{q}{2} \\ &\quad +2ibD\sin q\,\tfrac{\partial P}{\partial X_2}/c + bD\cos q\,\tfrac{\partial^2 P}{\partial X_1^2}/c.\end{aligned} \tag{13}$$

Finally, using the transformation $\xi_k = X_k - v_g T_k$ and $\tau_k = T_k$, we obtain

$$i\frac{\partial P}{\partial \tau_2} + \frac{l^*}{2}\frac{\partial^2 P}{\partial \xi_1^2} + m^* \mid P\mid^2 P + i\frac{n^*}{2}P = 0. \tag{14}$$

Eq. 14 presents the modified complex Ginzburg-Landau equation, that shows the evolution of traveling pulses in our considered diffusive network.

Remark 1 The real dispersion coefficients l^*, n^* are described as follows

$$l^* = \frac{cbD\omega^2\cos q - b^2 D^2 \sin^2 q}{c^2\omega^3},$$
$$n^* = (b/c - c) + 4D\sin^2(q/2)$$

Remark 2 Other complex dissipation coefficient m^* is given by

$$m^* = \frac{1}{2\omega}\left[i\omega c + \left(\frac{4}{d-b} - b \right.\right.$$
$$\left.\left. + \frac{2}{-4\omega^2 + d - b + (4bD\sin^2 q)/c}\right)\right].$$

m_r^* and m_i^* indicate real and imaginary terms of m^* respectively. The sign of $l^* m_r^*$ plays the role to find the modulational instability as the dispersion coefficient shows real quantity. Positive and negative values of $l^* m_r^*$ show that plane wavy profiles are unstable and stable [39]. This stability criterion is independent on the procedure in which the wave propagates. In the positive domain of $l^* m_r^*$, we can observe the propagating impulses for any wave carrier.

4 Traveling wave solution

In this section, we derive the solution of Eq. 14 to study the dynamics of the traveling wave profiles. We construct the solution of Eq. 14 in the following form [19,39]

$$P(\xi_k, \tau_2) = \frac{P_0 e^{\phi}}{1 + e^{(\phi+\phi^*)(1+i\beta)}}, \tag{15}$$

where $\phi = q\xi_k - \omega\tau_2$, $\beta = \gamma \pm \sqrt{2+\gamma^2}$ and $\gamma = \frac{3m_r^*}{2m_i^*}$. Substitute Eq. 15 in Eq. 14, we obtain

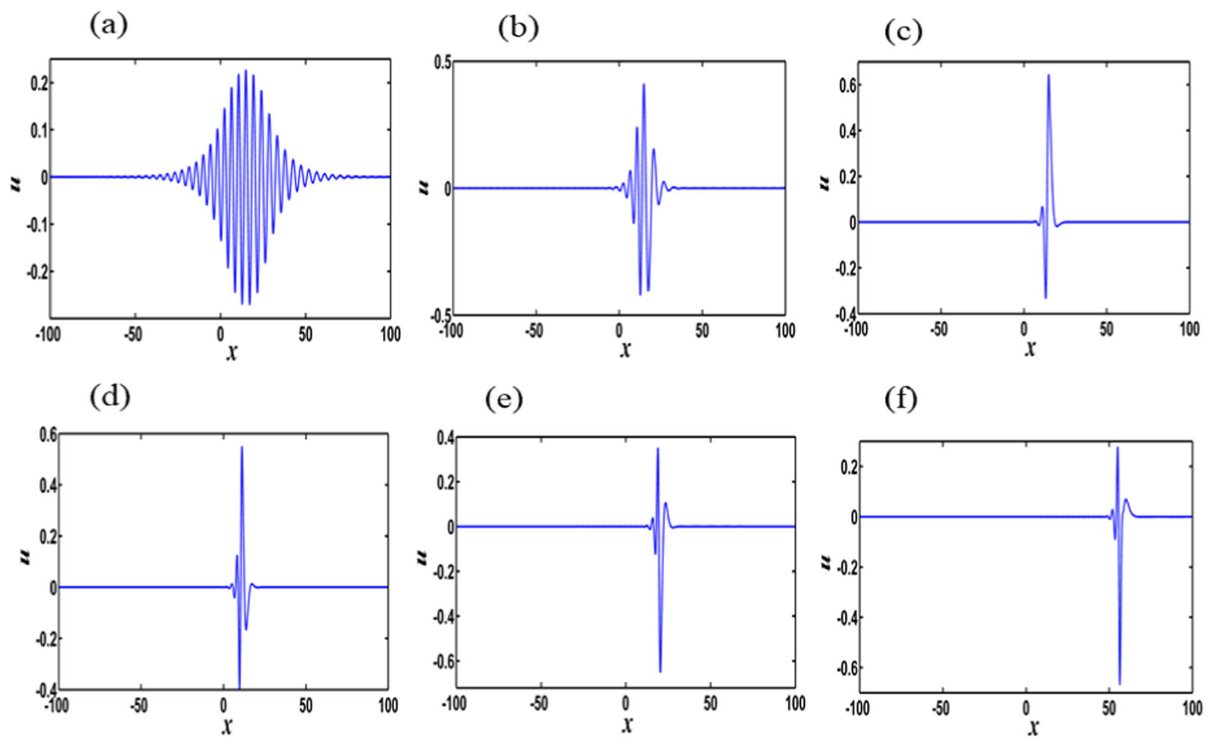

Fig. 2 Emergence of spatial activities of traveling waves for the diffusively coupled H-R system. Different formations of traveling pulses are shown. The multipulses and disappearance of multibumps with the emergence of solitary traveling waves are presented with the parameters $t = 10$, $c = 3$, $b = 5$, $d = 10$, $q = 1.5$, $P_0 = 1$ with the effects of varying ϵ and fixed $D = 0.01$ (a) $\epsilon = 0.25$, (b) $\epsilon = 0.45$, (c) $\epsilon = 0.7$, and various diffusion coefficients (d) $D = 0.01$, (e) $D = 1$, and (f) $D = 10$ with $\epsilon = 0.6$

$$P = P_0 \frac{e^{-\phi} + \cos 2\phi\beta e^{\phi}}{2(\cosh 2\phi + \cos 2\phi\beta)} - i\left(P_0 \frac{\sin 2\phi\beta e^{\phi}}{2(\cosh 2\phi + \cos 2\phi\beta)}\right). \tag{16}$$

From Eq. 7, we obtain

$$m = 2(P_r \cos\Omega - P_i \sin\Omega) + \epsilon[Q + 2(R_r \cos 2\Omega - R_i \sin 2\Omega)] + \mathcal{O}(\epsilon^2), \tag{17}$$

where R_r and R_i indicate real and complex parts of R. Finally, using Eqs. 16-17 and the expression $v_k = \epsilon m_k$, we obtain

$$\begin{aligned} u_k = {} & \varepsilon P_0 \left[\frac{\cos(\Omega_k - 2\beta\phi_k)e^{\phi_k} + e^{-\phi_k}\cos\Omega_k}{\cosh 2\phi_k + \cos 2\beta\phi_k}\right] \\ & + \varepsilon P_0^2 \left[-\frac{1}{(d-b)(\cosh 2\phi_k + \cos 2\beta\phi_k)}\right] \\ & + \varepsilon^2 P_0^2 \frac{(e^{-2\phi_k} + 2\cos\beta\phi_k + e^{2\phi_k}\cos 4\beta\phi_k)a_1 \cos 2\Omega_k}{2(\cosh 2\phi_k + \cos 2\beta\phi_k)^2} \\ & - \varepsilon^2 P_0^2 \frac{(2\sin 2\beta\phi_k + e^{2\phi_k}\sin 4\beta\phi_k)a_1 \sin 2\Omega_k}{2(\cosh 2\phi_k + \cos 2\beta\phi_k)^2}. \end{aligned} \tag{18}$$

In Fig. 2, we observe the impact of the series expansion perturbation parameter, ϵ on the traveling waves for fixed values of remaining parameters in the network system. The constants P_0 and q take small values [19]. The variations in the parameters do not affect the dynamical behavior significantly. The parameter q is considered in a small regime mentioned in Fig. 2(a). At lower perturbation value $\epsilon = 0.25$, we obtain the envelope of multipulses for a fixed diffusive coupling strength $D = 0.01$, where the bumps are unstable and transient [20]. As $\epsilon = 0.45$ increases, there is no significant activity to generate new multibumps and the number of multipulses decrease (see Fig. 2(b)). Then, at higher $\epsilon = 0.7$, the impulse is localized (see Fig. 2(c)) in the extended network, it shows a single solitary pulse. As the value of ϵ increases, the amplitude of the traveling pulses increases. Then more ions travel across the membrane, indicating that high-amplitude action potentials will be generated. This type of phenomenon can be occurred by the fluctuations in neuronal firings [10,19]. ϵ effects on the structural patterns of the wave

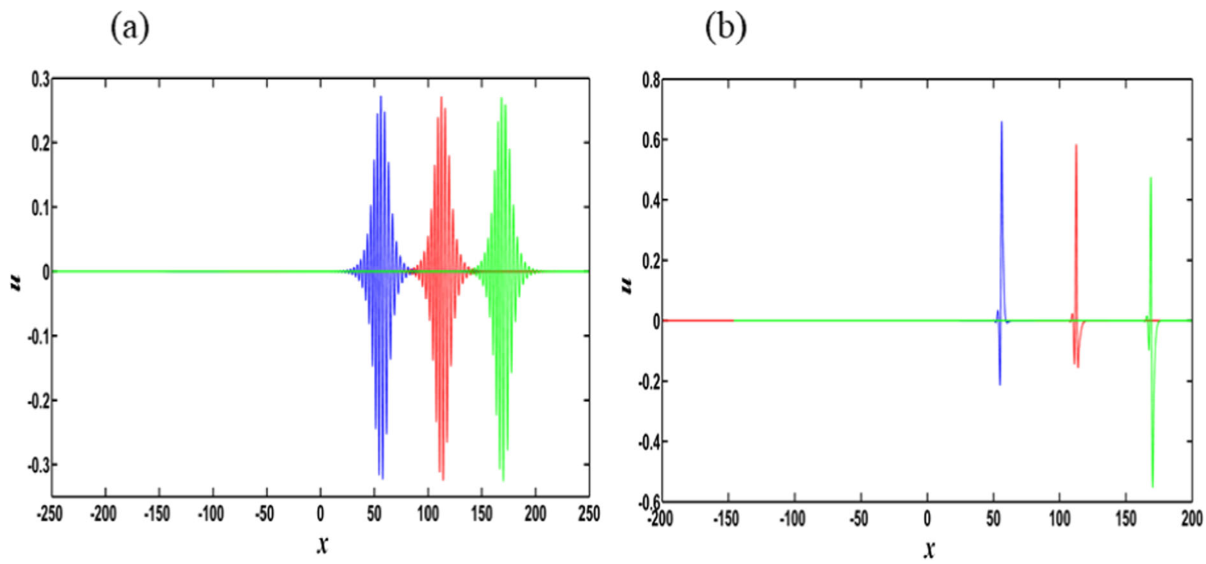

Fig. 3 Multipulses and single solitary traveling wave impulses at various time period. The wave profiles appear in the form of symmetric way, however they show delayed over time with same amplitude modulations, the parameters are as follows $c = 3$, $b = 5$, $d = 10$, $D = 0.01$, $q = 1.5$, $P_0 = 1$, (a) $\epsilon = 0.3$ and (b) $\epsilon = 0.75$ ($t = 50$ (blue), $t = 100$ (red), $t = 150$ (green))

profiles and transforms the pattern into single solitary pulses. However, the feature of the traveling pulses changes with time domain when the value of ϵ is fixed and we change the diffusive coupling strength, D. This exhibits simple traveling pulses and with the change of the values of D, the amplitudes of the single propagating waves decrease as shown in Fig. 2(d)-(f). We show how the multipulses come out with neighboring standing pulses that collapse to single-traveling pulse. The solitary wavy pulse corresponds to the multipulses, in which individual bumps are transient and nonpropagating. We show the changes of the single and multipulses changing with time t in Fig. 3. The amplitude of the envelope of multipulses is independent of time. The formations and amplitude of the asymmetric traveling waves change its features with time, therefore it is structurally unstable.

5 Instability of plane waves

In the previous section, envelope of a single and multi-bump wave profiles are observed for Eq. 14. Next, we find stability conditions of plane waves for small perturbations. Amplitude modulated pulses appear with the effects of instability of plane waves. We consider a plane waves in the following form

$$P\,(\xi_k, \tau_2) = P_0\, e^{i\,(\alpha\,\xi_k\, - l\,\tau_2)}, \tag{19}$$

where, P_0, l and α indicate the plane wave amplitude, angular frequency and the wave number, respectively. Using Eq. 19 in Eq. 14 and equating the real and imaginary terms, we obtain

$$l = \alpha^2\, \frac{l^*}{2} - m_r^*\, P_0^2, \tag{20}$$

and

$$m_i^*\, P_0^2 + \frac{n^*}{2} = 0. \tag{21}$$

Eq. 20 describes the dispersion relation of plane waves. Now, we investigate the instability of the plane wave and assume a solution in the following form

$$P\,(\xi_k, \tau_2) = (P_0 + b_1\,(\xi_k, \tau_2))\, e^{i\,(\alpha\,\xi_k\, - l\,\tau_2 + b_2(\xi_k, \tau_2))}, \tag{22}$$

where, the perturbation amplitude $b_1(\xi_k; \tau_2)$ is very small with respect to P_0. Using Eqs. 14, 22 and neglecting the nonlinear terms of perturbations b_1 and b_2, we obtain the following equations

$$-\,P_0\, b_{2_{\tau_2}} + \frac{l^*}{2}\, b_{1_{\xi\xi}} - P_0\, l^*\, \alpha\, b_{2_\xi} + 2\, P_0^2\, m_r^*\, b_1 = 0, \tag{23}$$

$$b_{1_{\tau_2}} + \frac{l^*}{2} P_0\, b_{2_{\xi\xi}} + l^*\,\alpha\, b_{1_\xi} - n^*\, b_1 = 0. \tag{24}$$

Eqs. 23–24 are known as the evolution of the perturbation. We assume the solutions of the above Eqs. 23 – 24 in the form

$$b_1 = b_{10}\, e^{i(\gamma_1\,\xi - \gamma_2\,\tau_2)} + c.c., \tag{25}$$

$$b_2 = b_{20}\, e^{i(\gamma_1\,\xi - \gamma_2\,\tau_2)} + c.c., \tag{26}$$

where, γ_1 and γ_2 indicate wave number of the perturbation and propagation frequency, respectively. It is noted that, the wave number shows real quantity and the propagation frequency is complex. Here, c.c. denotes complex conjugate.

Now, substituting the solutions Eqs. 25–26 in Eq. 23 and Eq. 24, we consider the following linear homogeneous equations for b_{10} and b_{20}

$$\left(2\, P_0^2\, m_r^* - \frac{l^*}{2}\,\gamma_1^2\right) b_{10} + i\, P_0\,(\gamma_2 - l^*\,\alpha\,\gamma_1)\, b_{20} = 0, \tag{27}$$

$$\left(-n^* - i\,(\gamma_2 - l^*\,\alpha\,\gamma_1)\right) b_{10} - \frac{l^*}{2}\, P_0\,\gamma_1^2\, b_{20} = 0, \tag{28}$$

that can be described in the matrix notation as

$M M_1 = 0$

where,

$$M = \begin{pmatrix} 2\, P_0^2\, m_r^* - \frac{l^*}{2}\,\gamma_1^2 & i\, P_0\,(\gamma_2 - l^*\,\alpha\,\gamma_1) \\ -n^* - i\,(\gamma_2 - l^*\,\alpha\,\gamma_1) & -\frac{l^*}{2}\, P_0\,\gamma_1^2 \end{pmatrix}$$

and

$$M_1 = \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix}.$$

If M represents a singular matrix, i.e., det(M)=0, Eqs. 27–28 are the nontrivial solutions, from which it results that

$$(\gamma_2 - l^*\,\alpha\,\gamma_1)^2 = \frac{l^{*2}}{4}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right) + i\, n^*\,(\gamma_2 - l^*\,\alpha\,\gamma_1)\,. \tag{29}$$

If $s = \gamma_2 - l^*\,\alpha\,\gamma_1$, Eq. 29 becomes

$$s^2 - i\, n^*\, s - \frac{l^{*2}}{4}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right) = 0. \tag{30}$$

Equation 29 describes the dispersion relation of the perturbation. It is noted that, for a fixed value of γ_1, the term $\frac{m_r^*}{l^*}$ plays an important role in controlling the dynamics of γ_2. Solving Eq. 30, we obtain

$$s = i\,\frac{n^*}{2} \pm \frac{\sqrt{l^{*2}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right) - n^{*2}}}{2}. \tag{31}$$

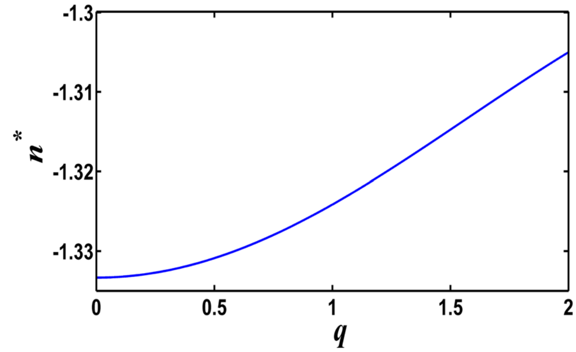

Fig. 4 Effects of wave vector q on the real dissipation coefficient n^* for $D = 0.01$

The following three cases may arise according to the sign of the discriminant:

Case 1: $l^{*2}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right) - n^{*2} = 0$, i.e., $s = i\,\frac{n^*}{2}$.

The imaginary part of γ_2 i.e., imaginary part of s, is equal to $n^*/2$. Fig. 4 shows that $n^* < 0$, which measures that the wave propagates around its original value and the perturbations collapsed after a certain time duration. So, the plane wave is stable.

Case 2: $l^{*2}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right) - n^{*2} > 0$, i.e., $Im(s) = i\,\frac{n^*}{2}$.

Similarly it can be deduced that the plane wave is also stable.

Case 3: $l^{*2}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right) - n^{*2} < 0$, i.e.,

$$s = i\left(\frac{n^*}{2} \pm \frac{\sqrt{n^{*2} - l^{*2}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right)}}{2}\right).$$

We also know that, if $Im(s) < 0$, the plane wave is stable. This implies

$$l^{*2}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right) > 0.$$

The above expression indicates that, if l^* and m_r^* have opposite signs, i.e. $l^* m_r^* < 0$, the plane wave solution is stable. Similarly, it is shown that the plane wave solution exists in unstable region, i.e., the perturbations increase exponentially in time, if $l^* m_r^* > 0$. The local growth rate, i.e., the gain for modulational instability can be written as

$$F = |\mathrm{Im}\, y| = |\mathrm{Im}\, \gamma_2| \\ = \frac{1}{2}\left(n^* + \sqrt{n^{*2} - l^{*2}\,\gamma_1^2\left(\gamma_1^2 - \frac{4\,P_0^2\,m_r^*}{l^*}\right)}\right). \tag{32}$$

If the wave number $\gamma_1 = P_0\sqrt{\frac{m_r^*}{l^*}}$ and the corresponding gain $F = \frac{1}{2}\left(n^* + \sqrt{n^{*2} + 3\,P_0^4\,m_r^{*2}}\right)$, the gain becomes maximum, i.e., the plane wave modulates itself. The plane wave amplitude and the dissipation coefficient of the CGLE system (Eq. 14) show that it controls the growth rate of modulational instability which is clear from the above expression of F.

6 Conclusions

To obtain and compute the traveling pulses solution describing the motion of impulses, we described the characteristics of the localized nonlinear excitations in a diffusively connected 2D H-R type model network characterized with nearest neighbour interactions. We transform the coupled dynamical equations into wave form and proceed to analyze the dynamical features with traveling pulses solution. To achieve our results, we derive multiple scale analysis method with semi discrete approximation. We obtain the dispersion and group velocity equations with diffusive coupling strengths and other system parameters. Then, we derive the modified CGLE equation from the original dynamical equations representing the equations of motion that governs the evolution of the traveling waves solution. The analytical results demonstrate envelope solitary pulse and multipulses that are found in numerical simulations. The single traveling pulses of the biophysically motivated extended excitable system are demonstrated from nonlinear dynamical system's perspective. Successive transitions from single traveling pulses to multipulses are observed. The modulated travelling pulses and its features are obtained which are influenced by diffusive couplings and small series expansion parameter.

Our observed results may be interesting to investigate further various characteristics of traveling wave profiles in spatial excitable media [19,20,39]. The electrically coupled neurons in this type of network with weak coupling can communicate and participate in the collective neural dynamics. The neurons can also effectively participate in information processing in spatial domain [7,34,39,41]. These pulses and traveling waves are in fact the effects of catalyst which allows the synapses to exchange neuro-transmitters. The generated wave somehow releases the neuro-transmitters, within the synapse, from one neuron to another. Neuronal ensembles can process stimuli in various ways. The propagating waves exhibit nonlinear-envelope solitons in their amplitudes, as we showed in this study. The amplitude of the traveling wave impulses depend on the series expansion parameter and coupling that indicate the variations in ionic movements across neuronal membrane, that leads to the generation of various amplitude's action potentials. This is caused by fluctuations in the sequence of neuronal firing times [33,34]. The propagation of impulses in the extended spatial excitable medium depend on the transmembrane voltage differences. The appearance of emerging traveling wave propagation can be explored further to measure different characteristics in the coupled network of biophysically excitable system by the variations of system parameters and diffusive coupling. The results show that the network can effectively participate in the collective behavior in both space-time scales.

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Declarations

Conflict of interest The authors declare that they have no conflicts to disclose regarding the publication of this article.

Ethical standard The authors state that this research complies with ethical standards.

Appendix

Equation 6 can be written in the following form

$$\left(\epsilon^2\frac{\partial^2 P}{\partial T_1^2} - \epsilon 2i\omega\frac{\partial P}{\partial T_1} - \epsilon^2 2i\omega\frac{\partial P}{\partial T_2} - \omega^2 P\right)e^{i\,\Omega_k} \\ + \left(\epsilon^2\frac{\partial^2 P^*}{\partial T_1^2} + \epsilon 2i\omega\frac{\partial P^*}{\partial T_1} + \epsilon^2 2i\omega\frac{\partial P^*}{\partial T_2} - \omega^2 P^*\right)$$

$$
\begin{aligned}
& e^{-i\,\Omega_k}+\left(-\epsilon^2\,4i\omega\frac{\partial R}{\partial T_1}-\epsilon 4\,\omega^2\,R\right)e^{2i\,\Omega_k} \\
& +\left(\epsilon^2\,4i\omega\frac{\partial R^*}{\partial T_1}-\epsilon 4\,\omega^2\,R^*\right)e^{-2i\,\Omega_k} \\
& +\epsilon^2\left[\left(\frac{b}{c}-c\right)+c\left(P^2\,e^{2i\,\Omega_k}+P^{*2}\,e^{-2i\,\Omega_k}+2P\,P^*\right)\right] \\
& \left(-i\omega P\,e^{i\,\Omega_k}+i\omega\,P^*\,e^{-i\,\Omega_k}\right)+(d-b) \\
& \left[P\,e^{i\,\Omega_k}+P^*\,e^{-i\,\Omega_k}+\epsilon\left(Q+R\,e^{2i\,\Omega_k}+R^*\,e^{-2i\,\Omega_k}\right)\right] \\
& +\epsilon\left(P^2\,e^{2i\,\Omega_k}+P^{*2}\,e^{-2i\,\Omega_k}+2P\,P^*\right) \\
& +2\epsilon^2\left(P\,e^{i\,\Omega_k}+P^*\,e^{-i\,\Omega_k}\right)\left(Q+R\,e^{2i\,\Omega_k}+R^*\,e^{-2i\,\Omega_k}\right) \\
& +\frac{b}{3}\,\epsilon^2\left(P^3\,e^{3i\,\Omega_k}+P^{*3}\,e^{-3i\,\Omega_k}\right. \\
& \left.+3\,P^2\,P^*\,e^{i\,\Omega_k}+3P\,P^{*2}\,e^{-2i\,\Omega_k}\right)+o\left(\epsilon^3\right) \\
= & \frac{b}{c}D\,(\left(P+\epsilon\frac{\partial P}{\partial X_1}+\epsilon^2\,\frac{\partial P}{\partial X_2}+\frac{\epsilon^2}{2}\frac{\partial^2\,P}{\partial X_1^2}\right)e^{iq}\,e^{i\,\Omega_k} \\
& +\left(P^*+\epsilon\frac{\partial\,P^*}{\partial X_1}+\epsilon^2\,\frac{\partial\,P^*}{\partial X_2}+\frac{\epsilon^2}{2}\frac{\partial^2\,P^*}{\partial X_1^2}\right)e^{-iq}\,e^{-i\,\Omega_k} \\
& +\epsilon\left(Q+\epsilon\frac{\partial Q}{\partial X_1}+\left(R+\epsilon\frac{\partial R}{\partial X_1}\right)e^{2iq}\,e^{2i\,\Omega_k}\right. \\
& \left.+\left(R^*+\epsilon\frac{\partial\,R^*}{\partial X_1}\right)e^{-2iq}\,e^{-2i\,\Omega_k}\right) \\
& -2\left(P\,e^{i\,\Omega_k}+P^*\,e^{-i\,\Omega_k}+\epsilon\left(Q+R\,e^{2i\,\Omega_k}+R^*\,e^{-2i\,\Omega_k}\right)\right) \\
& +\left(P-\epsilon\frac{\partial P}{\partial X_1}-\epsilon^2\,\frac{\partial P}{\partial X_2}+\frac{\epsilon^2}{2}\frac{\partial^2\,P}{\partial X_1^2}\right)e^{-iq}\,e^{i\,\Omega_k} \\
& +\left(P^*-\epsilon\frac{\partial\,P^*}{\partial X_1}-\epsilon^2\,\frac{\partial\,P^*}{\partial X_2}+\frac{\epsilon^2}{2}\frac{\partial^2\,P^*}{\partial X_1^2}\right)e^{iq}\,e^{-i\,\Omega_k} \\
& +\epsilon\left(Q-\epsilon\frac{\partial Q}{\partial X_1}+\left(R-\epsilon\frac{\partial R}{\partial X_1}\right)e^{-2iq}\,e^{2i\,\Omega_k}\right. \\
& \left.\left.+\left(R^*-\epsilon\frac{\partial\,R^*}{\partial X_1}\right)e^{2iq}\,e^{-2i\,\Omega_k}\right)\right) \\
& +i\omega\,\epsilon^2\,D(-P\,e^{iq}\,e^{i\,\Omega_k}+P^*\,e^{-iq}\,e^{-i\,\Omega_k} \\
& -P\,e^{-iq}\,e^{i\,\Omega_k}+P^*\,e^{iq}\,e^{-i\,\Omega_k}) \\
& -2i\omega\,\epsilon^2\,D\left(-P\,e^{i\,\Omega_k}+P^*\,e^{-i\,\Omega_k}\right)+o\left(\epsilon^3\right),
\end{aligned}
$$

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